Autoparametric Resonance in a Spring-Pendulum System

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Part I
Preliminary Mathematics

1 Formulating Ideas

1.1 Initial Plans

Initially, we intended to investigate the motion of double pendulums, after some brief research into the topic, we discovered that the mathematics that defines these systems was far too advanced for us to understand at this point in time, or to learn within the time frame we'd been given to devise our experiment.

Following this realisation, we debated looking into the motion of bifilar pendulum systems, which involve two pendulums connected by a rigid rod. However, we ultimately decided that this idea was somewhat uninspired and did not interest us to a massive extent.

1.2 Final Plans

I happened upon Autoparametric Resonance (a unique behaviour in spring-pendulum systems) through a video on Youtube produced by science and engineering content creator, Steve Mould (youtu.be/MUJmKl7QfDU), showcasing the phenomenon at a basic level, as his content is aimed towards the general audience; this, unfortunately, meant very little mathematics was discussed.

After excitedly sharing this video with the group, followed by some brief research into the topic, we decided that this was what we wanted to investigate for our Physics Project, for a variety of reasons: It was a feasible experiment to carry out with the equipment available to us at school, it had not been heavily researched and we had a lot of questions that we wished to answer, and there were many ways to link in with Maths and Computer Science (which was a requirement for the project).

1.3 What is Autoparametric Resonance?

Autoparametric resonance is a rare, phenomenon that occurs with spring-pendulum systems when certain parameters are met. The effect is characterised by the periodic oscillation of the system between two behaviours; behaving like a simple pendulum, and behaving like a Spring-Mass system.

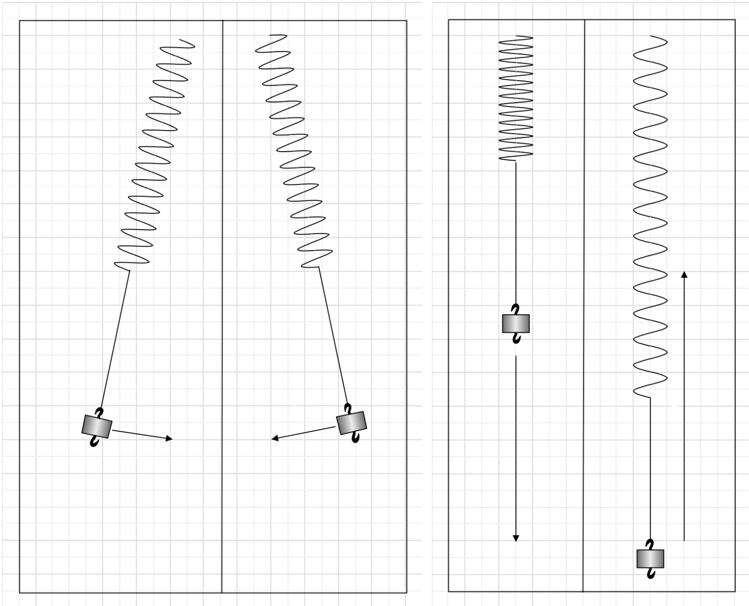

Figure 1: Oscillating Behaviours of Spring-Pendulum System

2 Deriving the "Autoparametric Formula"

2.1 "Autoparametric Formula"

As Autoparametric Resonance (APR) is a parametric behaviour, it only occurs, when certain parameters are met. Therefore a set of rules or formulae is necessary to define these parameters; I have dubbed this the Autoparametric Formula.

However, we faced a clear issue that, we do not know this formula, and could not find one through research online.

The only fact we knew was an offhand comment from the original video by Steve Mould stating that

$$L_0 = 3\Delta L$$

Where L_0 is the total initial length of the spring-pendulum system, and ΔL is the maximum extension after releasing the mass.

I was not keen on just accepting Steve Mould's words as fact, when he showed no mathematical proof that it was true. Therefore, I embarked on deriving my own formula.

2.2 Prerequisite Knowledge

In order to prove the Autoparametric Formula, a number of facts need to be known and accepted, these are:

$$F = k\Delta L$$

Where F is force applied to spring, k is the spring constant of the spring, and ΔL is the maximum extension of the spring.

$$T_p = 2\pi\sqrt{\frac{L}{g}}$$

Where T_p is the time period of a simple pendulum, L is the length of the pendulum, and g is acceleration due to gravity.

$$T_s = 2\pi\sqrt{\frac{m}{k}}$$

Where T_s is the time period of a spring, m is the mass attached to the end of the spring, and k is the spring constant.

$$T_p = 2T_s$$

This is because, for the spring and pendulum to resonate, the springs compression must add energy to the pendulums swing at exactly the peak of its swing, on both sides, therefore the spring has to complete 2 periods for every 1 of the pendulum.

2.3 Derivation

Using Hooke's Law we can solve for ΔL in terms of m and k:

$$F = k\Delta L$$

$$\therefore mg = k\Delta L$$

$$\therefore \Delta L = \frac{mg}{k}$$

If we define L_0 as the initial system length, then we can say that $L = L_0 + \Delta L$, and use this face to alter the formula for T_p slightly:

$$T_p = 2\pi\sqrt{\frac{L}{g}}$$

$$\therefore T_p = 2\pi\sqrt{\frac{L_0 + \Delta L}{g}}$$

Now, we can apply our formulas for T_p and T_s into the equation $T_p = 2T_s$:

$$2\pi\sqrt{\frac{L_0 + \Delta L}{g}} = 2(2\pi\sqrt{\frac{m}{k}})$$

$$\therefore \sqrt{\frac{L_0 + \Delta L}{g}} = 2\sqrt{\frac{m}{k}}$$

$$\therefore \frac{L_0}{g} + \frac{\Delta L}{g} = 4\frac{m}{k}$$

$$\therefore \frac{L_0}{g} + \frac{\frac{mg}{k}}{g} = 4\frac{m}{k}$$

$$\therefore \frac{L_0}{g} + \frac{m}{k} = 4\frac{m}{k}$$

$$\therefore \frac{L_0}{g} = 4\frac{m}{k} - \frac{m}{k}$$

$$\therefore \frac{L_0}{g} = 3\frac{m}{k}$$

$$\therefore L_0 = 3g\frac{m}{k} \Leftrightarrow L_0 = 3\Delta L$$

This confirms Steve's initial statement as true.

Part II
Preliminary Experimentation

3 Decision Making

3.1 Determining Variables

After calculating the necessary formula needed to continue with the investigation, we realised that we had 3 variables involved, and needed to eliminate one in order to conduct a proper experiment. We elected to keep the spring constant as a control variable, as it would be the most difficult to increment and measure. This meant we only needed one spring for the experiment, the spring constant of which had to be calculated, in order to proceed.

4 Calculating the Spring Constant

4.1 Method

In order to calculate the spring constant of the spring that we used in our investigation, we made use of hooke's law. We attached our spring to a clamp and added a mass to the end, we incremented the mass, M, by 0.01kg and measured the extension in the spring, ΔL, then, we graphed these using Desmos' linear regression algorithm to calculate a spring constant value.

4.2 Data

Here is the data collected from that experiment:

M / kg	L / m	ΔL / m
0.00	0.123	0.000
0.01	0.127	0.004
0.02	0.132	0.009
0.03	0.136	0.013
0.04	0.141	0.018
0.05	0.145	0.022
0.06	0.148	0.025
0.07	0.152	0.029
0.08	0.156	0.033
0.09	0.160	0.037
0.10	0.164	0.041

After plotting Mg against ΔL in Desmos (taking $g = 9.81$), we used it's linear regression algorithm to find a line of best fit for the data, which gave us a line with of 24 which means our spring constant, $k = 24$.

Part III
Primary Experimentation

5 Method

5.1 Variables

Defining the variables within the experiment posed a challenge as the behaviour we are investigating is 'parametric' meaning certain parameters need to be met in order for it to occur within the system, we needed to stay within those parameters to carry out our experiment. To do this we had to consider the length (Initial length of the system, L_0) as a function of the mass (mass attached to end of string, m); meaning, even though we had to change the length of the system incrementally, we did so according to the formula $L_0 = 3g\frac{m}{k}$, therefore, the length is not a true independent variable.

The variables within our experiment were:

- Independent: $\begin{bmatrix} m \\ L_0(m) \end{bmatrix}$, The mass, and the length of the system as a function of the mass.
- Dependent: T, The time period of the Resonant Behaviour
- Control:
  1. Spring Constant
  2. Force of gravity
  3. Initial length of the Spring, (did not pass its elastic limit and deform)

5.2 Setup

To set up the experiment, we fastened the spring to the clamp using tape to minimise translation and rotation of the spring. Then, we tied a string of variable length to the spring and attached our mass to that string.

We also placed a stopwatch within the frame of our camera, in order to get an accurate reading for the time period during data analysis.

5.3 Iterative Method

In order to carry out the experiment, we attached a measured mass to a length of string, calculated using the Autoparametric Formula, and released it from rest (No extension in spring, and no angle in pendulum) and recorded one whole period of oscillation, which we defined as the time between the initial maximum peak of the pendulum's swing and the subsequent maximum peak.

We measured this by looking at the stopwatch that we setup within the frame of the video, reading the times of the pendulum peaks, then subtracting the difference. We repeated this 3 times for each data point and calculated the mean time period, before increasing the mass (and string accordingly) in order to record more measurements.

Part IV
Analysis of Results

6 Data

6.1 Table of Results

Below is the table of results for our primary experiment, with Independent Variables on the left hand side, and Dependent on the right:

m / g	L_0 / cm	T_1	T_2	T_3	T_{mean} / s
250.0	30.7	6.70	9.27	7.33	7.77
270.0	33.1	6.99	10.39	6.92	8.10
300.0	36.8	10.66	9.35	8.03	9.35
350.0	42.9	8.58	8.99	10.21	9.26
400.0	49.1	9.18	8.35	8.77	8.77
450.0	55.2	8.12	8.93	8.86	8.64
500.0	61.3	8.47	9.38	9.50	9.12
530.0	65.0	11.55	9.79	10.57	10.64
550.0	67.4	13.58	14.4	10.70	12.89

It is clear from brief analysis alone, of the data, that there is a positive correlation between mass and the time period between behavioural oscillations, however, this is not a satisfying conclusion to draw.

6.2 Using the Results

After collecting the raw data from the experiment, the only task remaining was to analyse our results in such a way that we could draw a significant conclusion from them about the relationship between the mass and time period between behavioural oscillations. To do this, we used desmos' polynomial regression algorithms to model our data at varying polynomial degrees.

7 Statistical analysis

7.1 Polynomial Models

We used Desmos' polynomial regression algorithms to calculate 7 different Models of varying order (Models were calculated using our data converted in Standard International Units):

$$M_0(x) = 9.4$$

$$M_1(x) = 9.8x + 5.4$$

$$M_2(x) = 60.0x^2 - 38.1x + 14.3$$

$$M_3(x) = 1332.2x^3 - 1538.7x^2 + 578.4x - 61.7$$

$$M_4(x) = 3783.8x^4 - 4721.8x^3 + 2006.9x^2 - 320.8x + 21.5$$

$$M_5(x) = 14374.8x^5 - 24965.9x^4 + 17.855.6x^3 - 6686.3x^2 + 1319.0x - 99.7$$

$$M_6(x) = 523571x^6 - 1242200x^5 + 1213200x^4 - 622814x^3 + 176827x^2 - 26259x + 1599$$

7.2 Standard Deviation Test

My first idea on how to measure the effectiveness of the models, was to look at the standard deviation of the difference between observed data and modelled data. I used this formula:

$$\sigma_n = \sqrt{\frac{\sum_{i=1}^{9}(T(x_i) - M_n(x_i))^2}{|x|} - (\frac{\sum_{i=1}^{9} T(x_i) - M_n(x_i)}{|x|})^2}$$

Where n = The order of the polynomial. $T(x)$ = the mean observed Time Period of a given mass, x. x_i = the set of the masses used to measure time period within our experiment, indexed in with, at i. This test produced a standard deviation for each polynomial model:

Order	Standard Deviation
0	1.543
1	1.045
2	0.906
3	0.266
4	0.178
5	0.169
6	0.085

These results indicate a clear trend that, as you increase the degree of the polynomial of model that you test, the spread of the difference between observed data and modelled data for that degree. There is also a minor indication that order 3 Model is significant as the maximum value of $|\sigma_n - \sigma_{n-1}|$ is when $n = 3$.

7.3 Chi-Squared Testing

A common statistical method of evaluating the accuracy of a given model at predicting a set of data is the Chi-Squared test, which I employed to my models and results with the formula:

$$\chi^2_n = \sum_{i=1}^{9} \frac{(T(x_i) - M_n(x_i))^2}{M_n(x_i)}$$

Following the same notation as with the standard deviation tests. The table of results which this test produced is as follows:

Order	χ^2 Test
0	2.029
1	0.856
2	0.681
3	0.061
4	0.028
5	0.026
6	0.006

This set of results is also clearly indicative of the polynomial models becoming more accurate as their order increases. The chi-squared test result data set is also makes the difference in accuracy between the model order 3 and order 2 far more obvious, this can be numerically demonstrated by calculating all values of

$$\frac{\chi^2_{n-1}}{\chi^2_n}$$

for all valid n:

n	$\frac{\chi^2_{n-1}}{\chi^2_n}$
1	2.370
2	1.257
3	11.253
4	2.162
5	1.077
6	4.076

The value for this test at $n = 3$ clearly shows that the improvement between models is largest between model, order 2, and model, order 3.

8 Conclusion

In conclusion, after analysing polynomial models of various degrees to fit the data, and consider the nature of polynomial models of increasing degree to tend towards completely interpolating the modelled data set, the most logical model would be the lowest order model that has a sufficiently low result from the chi-squared test, sufficiency of this low result being characterised by a large difference in result between it and it's previous model ($\frac{\chi^2_{n-1}}{\chi^2_n}$).

Therefore, there is sufficient evidence to conclude that the model that fits the data set most logically, in part, due to the previous explanation, but also due to Occam's razor (When choosing between many equally sufficient options, choose the simplest one), would be a model of order $n = 3$.

This statistical conclusion allows us to draw the final conclusion that, when the parameters that allow for Autoparametric Resonance to occur, within a spring-pendulum system, hold true, the relationship between the magnitude of the mass attached to the end of the string and the time period of the Autoparametric oscillations between the two spring-pendulum behaviours, is cubic.