3D Invasion Percolation: Fractal Geometry and Self-Organised Criticality

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Plagiarism declaration

> This piece of work is a result of my own work and I have complied with the Department's guidance on multiple submissions and on the use of AI tools. Material from the work of others not involved in the project has been acknowledged, quotations and paraphrases suitably indicated, and all uses of AI tools have been declared.

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AI Usage Declaration

- "Grammarly was used to enhance the writing style, grammar, and spelling of the report throughout."
- "Google Gemini was used to troubleshoot the code used to produce figures 1 and 3"

Link to GitHub Codebase

All codes used to produce the analysis and results in this report can be found at the GitHub repository: https://github.com/TheoBains/Percolation-Computational-Mathematics-II.git.

1 Introduction and Motivation

Standard site percolation models static site occupation with a fixed probability p. While mathematically fundamental, this model struggles to describe dynamic "drainage" processes, such as oil displacing water in porous rock. These processes are not random but follow paths of least capillary resistance [1].

This extension investigates **Invasion Percolation (IP)** [3]. Unlike the static case, IP has no tuning parameter p; instead, the cluster grows by invading the weakest site on its boundary. This report simulates IP in **three dimensions** to determine the fractal dimension (d_f) of the resulting structures and to demonstrate *Self-Organised Criticality* (SOC), where the system naturally drives itself to the percolation threshold p_c without external tuning.

2 Computational Methodology

We model the porous medium as a lattice ($L \times L \times L$ in 3D) where each site i is assigned a random "resistance" $r_i \in [0, 1]$. The simulation proceeds via a "burst" dynamic:

1. **Initialisation:** A seed site at the centre is filled (oil); the lattice has random resistances.
2. **Update:** Empty neighbours of the occupied site form the *interface*.
3. **Selection & Invasion:** The interface site with the **global minimum** resistance r_{min} is occupied, and its neighbours are added to the interface.

Optimisation: A naive search for the minimum resistance is $O(N)$ per step ($O(N^2)$ total), which is prohibitive for large 3D clusters ($N > 10^5$). To address this, I implemented a **Min-Heap** (priority queue) to manage the boundary. The heap property ensures the lowest resistance site is always at the root, reducing insertion and retrieval complexity to $O(\log N)$. This improves the overall algorithmic complexity to $O(N \log N)$, allowing efficient simulation of $L = 100$ systems in seconds.

3 Fractal Geometry and Dimensional Analysis

Invasion clusters are fractal because the fluid seeks rare, low-resistance paths, leaving behind large "fjords" of trapped defending fluid. We characterise this using the Fractal Dimension d_f, related to the cluster mass M and radius of gyration R_g by the power law $M(R_g) \sim R_g^{d_f}$.

We simulated invasion percolation on 2D ($L = 400$) and 3D ($L = 150$) lattices. Figure 1 displays the scaling behaviour for a representative realisation. Repeating this simulation multiple times yielded consistent results, establishing the following experimental fractal dimensions:

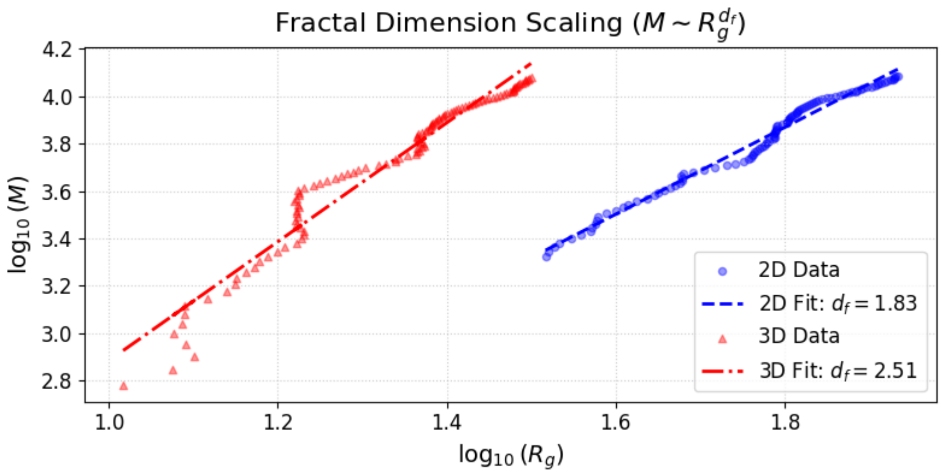

Figure 1: Scaling behaviour of invasion clusters. The linear fit ($M \sim R_g^{d_f}$) indicates fractal scaling with a distinct dimensionality jump from 2D to 3D.

- **2D Result:** $d_f \approx 1.88 \pm 0.02$. This aligns well with the theoretical literature value for invasion percolation with trapping, $d_f \approx 1.89$ [2].
- **3D Result:** $d_f \approx 2.52 \pm 0.03$. The increase confirms 3D clusters are significantly denser, as the extra degree of freedom allows the fluid to bypass obstacles.

4 Self-Organised Criticality (SOC)

A defining feature of IP is Self-Organised Criticality. In standard percolation, fractal structures only appear if p is manually tuned to p_c. In the invasion model, the system automatically evolves towards a critical state. We observed that the fluid almost exclusively invades sites with resistance $r < p_c$. Figure 2 displays the resistance histogram of invaded sites. The distribution is uniform up to a sharp cutoff near $p_c \approx 0.31$ (the known 3D threshold). This confirms the invasion process naturally identifies the critical spanning path.

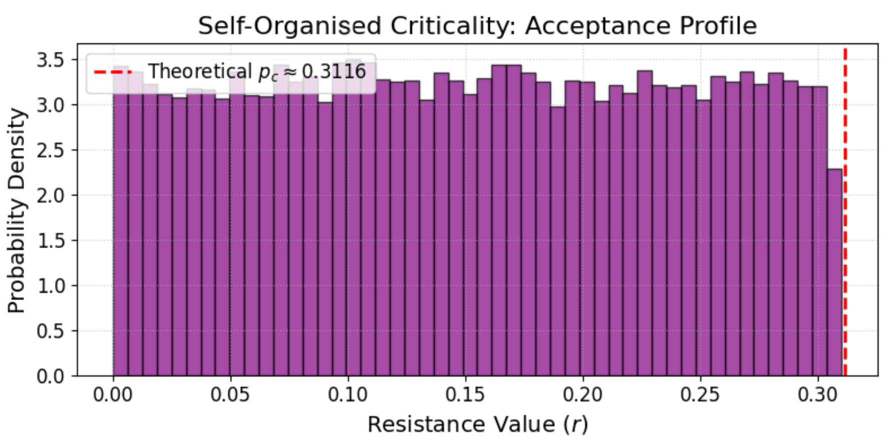

Figure 2: The 'Acceptance Profile' in 3D. The sharp drop-off near $p_c \approx 0.31$ confirms the system self-organises to the percolation threshold.

5 3D Visualisation

Figure 3 utilises voxel plotting to visualise the complex topology of the 3D cluster ($L = 150$), where colour represents the invasion time step (Blue=Early, Yellow=Late).

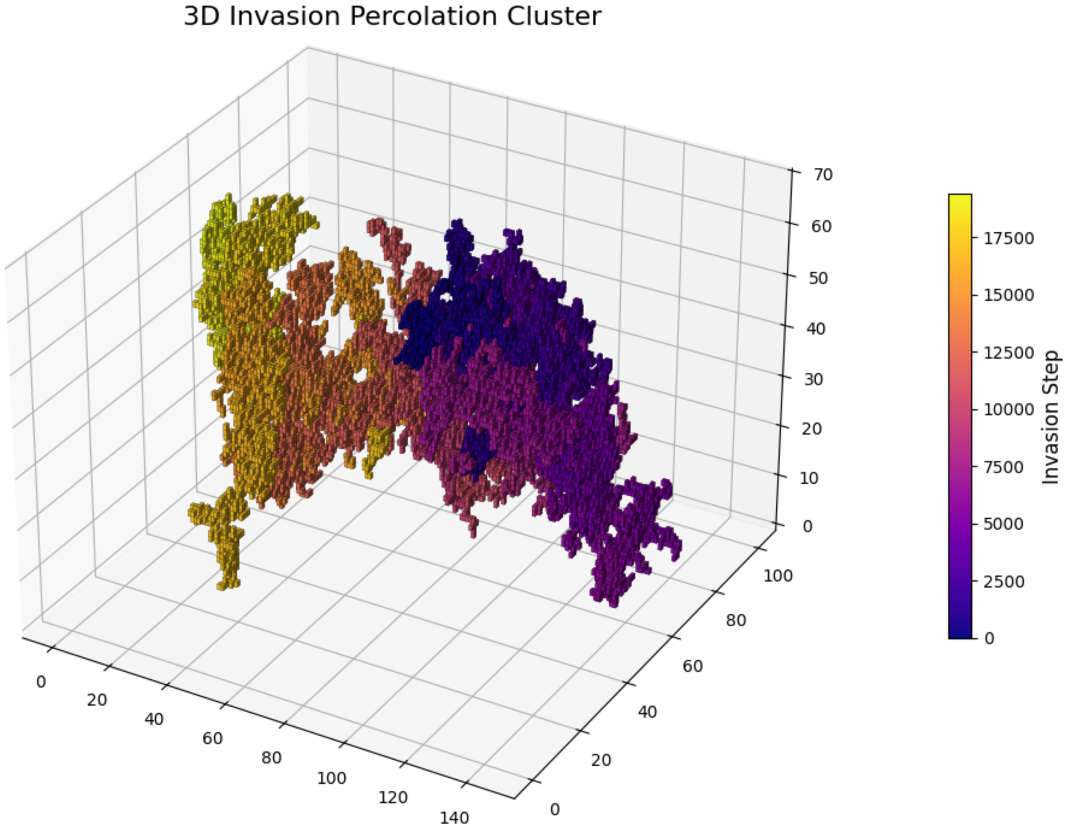

Figure 3: A 3D invasion cluster ($L = 150$). The structure is highly irregular, resembling coral, with large voids representing trapped defensive fluid.

The visualisation highlights the "bursty" nature of the flow. The fluid fills a local region (blue) before finding a narrow "throat" to break out into a new area (yellow/red). This creates a connected structure with massive holes—evidence of the trapping of the defending fluid, which cannot be captured in 2D cross-sections.

6 Conclusion

This project successfully simulated large-scale invasion percolation in 3D using a min-heap optimisation. We confirmed the fractal nature of the fluid displacement ($d_f \approx 2.52$) and demonstrated Self-Organised Criticality, showing how the system finds the threshold p_c without external tuning. These results provide insight into why oil recovery is often inefficient, as the fractal nature of the invasion leads to significant trapping of resources.

References

[1] A. Malthe-Sørenssen. Percolation theory using Python, volume 1029 of Lecture Notes in Physics. Springer, 2024.

[2] D. Stauffer and A. Aharony. Introduction to Percolation Theory. Taylor & Francis, 2 edition, 1994.

[3] D. Wilkinson and J. F. Willemsen. Invasion percolation. Journal of Physics A: Mathematical and General, 16(14):3365, 1983.