Finding Degenerate Points in Conics with Projective Geometry

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Contents

1 Introduction

In this paper, our main goal is to provide the reader with a fundamental understanding of how to find and identify degenerate points in conic sections with from minimal prerequisite knowledge.

To do this, we will begin by offering a brief but comprehensive overview of conic sections, including formal definitions, examples, and special cases. Subsequently, we look at the concept of degenerate points, exploring their definitions, examples, and practical applications. Following that, we will introduce the reader to some methods of finding and identifying degenerate points as well as giving a more in depth walk through of a method that makes use of projective geometry.

2 Conic Sections

2.1 Brief Overview

The study of conic sections explores the relationships between the curves formed from the intersections of cutting planes and a double cone.

A double cone is a cone which is symmetrical about the X-Y Plane and extends to infinity. It is most often defined by the formula

$$z^2 = x^2 + y^2$$

and looks as follows

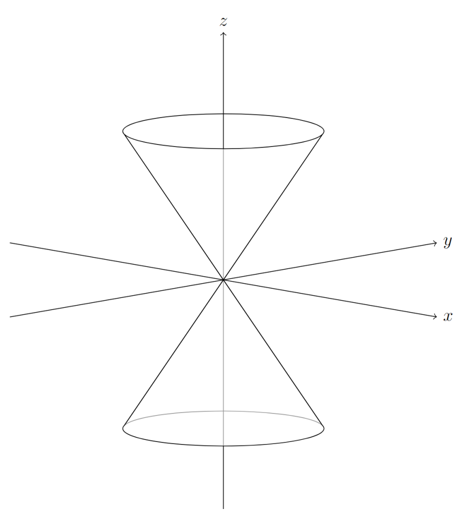

Figure 1: Double cone (a cone with two nappes)

The cutting plane is just a simple flat plane that is used to intersect with the double cone at different angles to create the conic sections.

There are 4 types of conic section (excluding degenerate cases): circles, ellipses, parabolas and hyperbolas. Fundamentally, they are all dependant on the angle that the cutting plane makes with the double cone.

Alternatively to the angle of the cutting plane, conic sections may also be defined purely in terms of plane geometry: it is the locus of all points P whose distance to a fixed point F (called the focus) is a constant multiple e (called the eccentricity) of the distance from P to a fixed line L (called the directrix). For $0 < \mathrm{e} < 1$ we obtain an ellipse, for $\mathrm{e} = 1$ a parabola, and for $\mathrm{e} > 1$ a hyperbola. A circle is a special case of an ellipse where the eccentricity is 0 and the foci are the same point, but its directrix can only be taken as the line at infinity.

There is a lot of interesting maths about conic sections to do with their foci, eccentricity and directrix but this section of the paper should only give the reader a brief understanding of what is required for the following sections on degenerate points.

All of these sections can be defined by a 2nd degree algebraic curve of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

2.2 Examples

Circles can be formed when the cutting plane is paralell with the X-Y plane and are just ellipses with an eccentricity of 0.

Circles are normally represented in the form

$$(x - h)^2 + (y - k)^2 = r^2$$

where r is the radius and the center of the circle is (h,k)

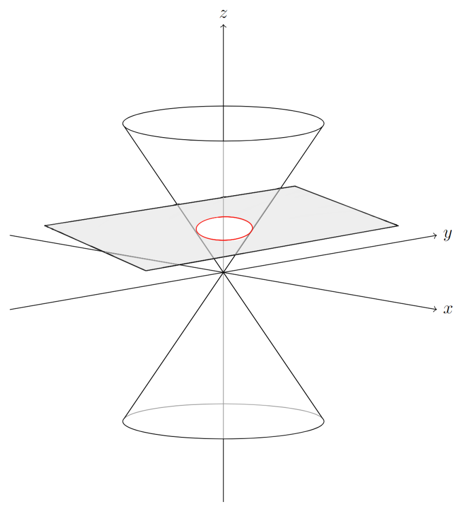

Figure 2: Circular Section

Ellipses are formed when the angle of the cutting plane is less than than that of the slope of the double cone.

Ellipses are normally represented in the form

$$\frac{(x-h)^2}{a} + \frac{(y-k)^2}{b} = 1$$

where the center is (h,k) and a and b change the foci of the ellipse.

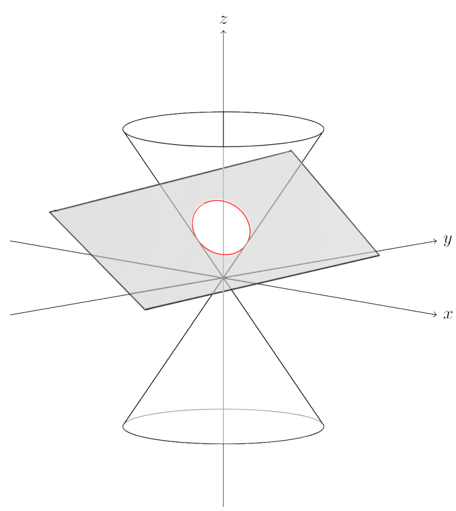

Figure 3: Elliptical Section)

Parabolas are formed when the cutting plane is paralell with the slope of the double cone.

Parabolas are normally represented in the form

$$y = a(x-h)^2 + k$$

where their vertex is at (h,k).

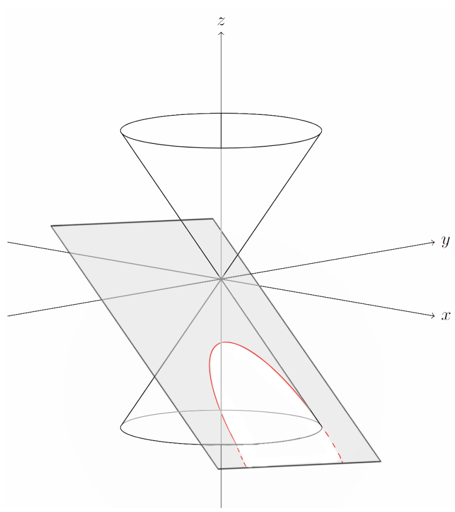

Figure 4: Parabolic Section

Hyperbolas are formed when the angle of the cutting plane is greater than that of the slope of the double cone.

Hyperbolas are normally represented in the form

$$\frac{(x-h)^2}{a} - \frac{(y-k)^2}{b} = 1$$

where the vertices are located at $(h \pm a, k)$ and the foci are located at $(h \pm \sqrt{a^2+b^2}, k)$

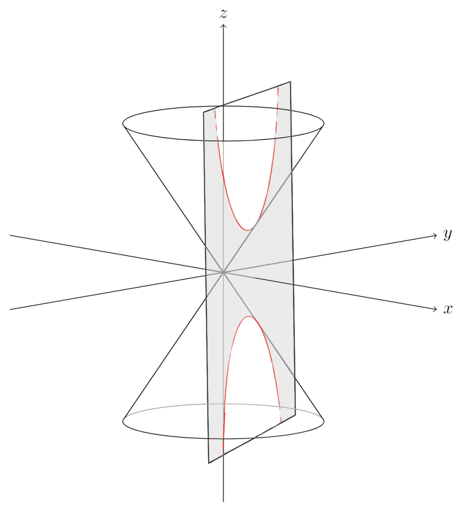

Figure 5: Hyperbolic Section

3 Degenerate Points in Conic Sections

3.1 Brief Introduction

There are 3 edge cases for the geometric interpretation I have just described. They are the degenerate conics.

All of these boundary cases occur when the cutting plane goes through the apex of the cone (the point where the 2 cones meet). Depending on the angle, the curve created can be a point, a line, or an X shape (intersecting lines).

There is a "lost" degenerate conic which you cannot create geometrically, this is two parallel lines. This can be imagined as a hyperbola with an eccentricity of ∞ . This case can only be created algebraically, not with an intersecting plane.

A degenerate point is considered either the point or the point where the X shape intersects itself. The case of the line does not have a degenerate point as it is just a line.

The degenerate cases and points have some unusual properties and can often represent key points when applied to physical situations. This makes locating them a valuable ability to have.

3.2 Definition

A formal definition for a degenerate conic is a conic (a second-degree plane curve, defined by a polynomial equation of degree two) that fails to be an irreducible curve. This means that the defining equation is factorable over the complex numbers (or more generally over an algebraically closed field) as the product of two linear polynomials.

A degenerate point refers to a point that lies on the conic but does not exhibit the typical properties associated with the conic sections. It can be identified as a point where the curvature or behavior of the curve changes abruptly, often resulting in singularities or points of inflection. This point may possess unique characteristics compared to other points on the conic, such as extreme tangential behavior, multiple intersections, or unusual geometric properties.

3.3 Examples

The case of a point is created when a conic is a circle or elipse with a radius of 0, creating the point. It is of the form

$$(x-h)^2 + (y-k)^2 = 0$$

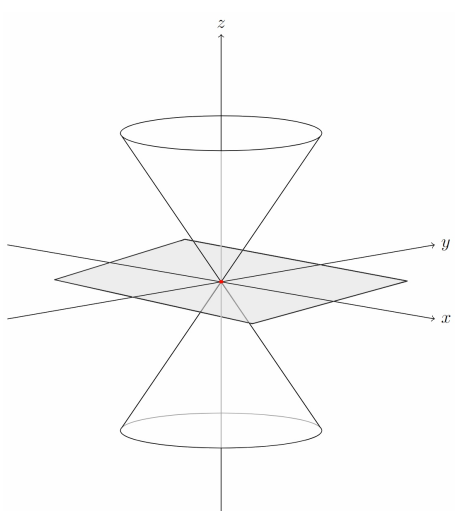

Figure 6: Degenerate Point

The case of the degenerate line is created when the cutting plane is parallel to the slope of the cone and it passes through the apex. This means that it will just touch the sides of the cone. The line is of the form

$$ax + by = 0$$

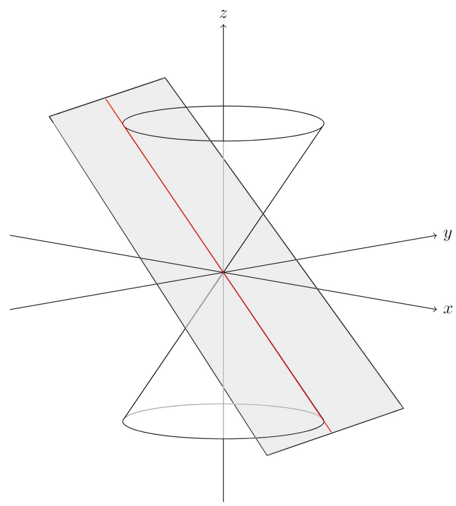

Figure 7: Degenerate Line

The case of the intersecting lines is a special case of the hyperbolas. It is of the form

$$\frac{(x-h)^2}{a} - \frac{(y-k)^2}{b} = 0$$

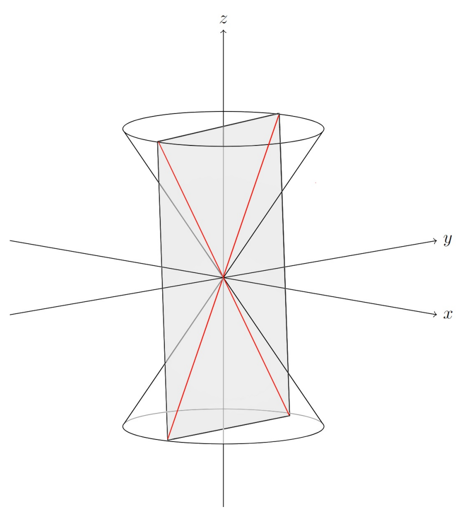

Figure 8: Degenerate Intersecting Lines

3.4 Uses

Degenerate points have some interesting and important uses in many fields. In maths, they are worthwhile researching and exploring as they have a direct link to poles within complex functions. This plays a vital role in Fermat's last theorem and the fact that these points have a one to one link demonstrates their importance in maths as the whole field of complex analysis relies on this link. They also have many non academic uses such as in lens and optical mechanics. As the shape of a lens can be defined by a conic, the degenerate point often have special or optimal properties. They are also used in celestial mechanics for orbits and stability analysis, gyroscopes for their models of motion, computer graphics for shape transformations and more.

4 Finding Degenerate Points

4.1 Issues to Overcome

It is simple to detect whether a conic is degenerate as it is one if it fails to be irreducible (can be factored into a product of 2 terms).

It is however harder to detect where these points are as they defy a lot of general geometry rules. We can however make use of their interesting property that they have no gradient.

4.2 Projective Geometry Method

This method makes use of the fact that the points do not have a gradient and aims to create surfaces with gradients that we can exploit to find their critical (degenerate) points.

We can start this method with our conic equation and check whether it is irreducible (this can be quite an annoying task and isn't necessary as we can perform a much easier check later on).

$$x^2 + 8x - y^2 + 4y + 12 = 0$$

$$(x + y + 2)(x - y + 6) = 0$$

Next, we will project this initial conic equation onto the plane $z = 1$. This will leave our conic hovering over the X-Y plane in its 2d form.

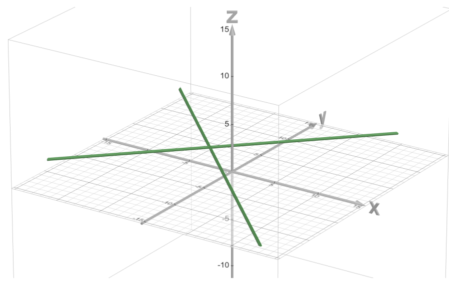

Figure 9: Conic projected onto z = 1

Now, we should imagine projecting an infinite amount of rays from the origin through the conic at $z = 1$. These rays will create an infinite series of enlargements of the initial conic which will then create a smooth surface of enlargements of the conic.

This sounds like a complex task but is actually pretty trivial as you just make the curve homogeneous by changing its form from

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

to the form

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0$$

This form creates the smooth surface

$$x^2 + 8xz - y^2 + 4yz + 12z^2 = 0$$

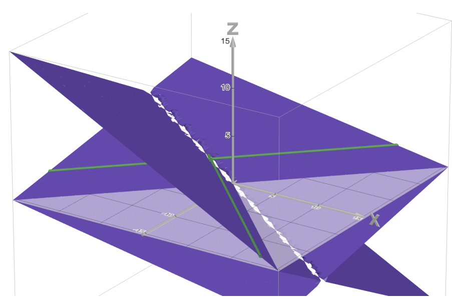

Figure 10: Smooth surface created by the rays

Now we are able to partially differentiate this new equation 3 times, with respect to x, y and z, giving us 3 new equations for the gradient of this surface.

Here is where we are able to perform the easier check for whether the conic is degenerate. We can do this by placing the coefficients from our partial derivatives (of the form $ax + by + cz$) into a 3x3 matrix and getting the determinant as such:

$$\begin{vmatrix} a_{x1} & b_{y1} & c_{z1} \\ a_{x2} & b_{y2} & c_{z2} \\ a_{x3} & b_{y3} & c_{z3} \end{vmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & 8 \\ 0 & -2 & 4 \\ 8 & 4 & 24 \end{bmatrix} \qquad \det(A) = 0$$

If the determinant is equal to 0, we can imply that the conic is degenerate. This is because of some interesting behaviours of differential homogeneous equations that I will not go into within this paper. Else, if the determinant is non zero, we can say that it is not degenerate.

Finally, to actually get the point, we can set these partial derivatives equal to 0 and solve in terms of z. This will give us a coordinate in the form (az, bz, z) . As we had the initial conic on the z=1 plane, we just sub z=1 back in and then get the coordinates of the degenerate point as if it was just in 2d again.

$$2x + 8z = 0$$

$$4z - 2y = 0$$

$$8x + 4y + 24z = 0$$

$$(-4z, 2z, z)$$

$$(-4,2)$$

5 Conclusion

In conclusion, degenerate points break a lot of rules in geometry but we can make use of these behaviours to locate them accurately and reliably with a method that makes strong use of projective geometry and planes. This method even has a link to matrices and homogeneous equations.

This is a very interesting topic and has a lot more to be explored than mentioned in this paper.